Exercise 59

A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x\to -1} f(x) = 2$. Evaluate

(a) $f(0)$ (b) $\lim_{x\to\infty} f(x)$

Solution

Construct the function f one step at a time. Since there's a vertical asymptote $x = 4$, place a factor of $(x - 4)$ in the denominator.

$$\frac{1}{x-4}$$

Since there's an x-intercept at $x = 1$, that means the function is zero at $x = 1$. Place a factor of $(x - 1)$ in the numerator.

$$\frac{x-1}{x-4}$$

Since there's a removable discontinuity (a hole) at $x = -1$, place a factor of $x + 1$ in the numerator and denominator.

$$\frac{(x-1)(x+1)}{(x-4)(x+1)}$$

Since there's a limit to be satisfied, include an arbitrary constant A in front of function.

$$f(x) = A\frac{(x-1)(x+1)}{(x-4)(x+1)}$$

Evaluate the limit of $f(x)$ as $x \to -1$.

$$\begin{aligned} 2 &= \lim_{x\to -1} f(x) \\ &= \lim_{x\to -1} A\frac{(x-1)(x+1)}{(x-4)(x+1)} \\ &= A \lim_{x\to -1} \frac{x-1}{x-4} \\ &= A\left(\frac{-2}{-5}\right) \\ &= A\left(\frac{2}{5}\right) \end{aligned}$$

Solve for A.

$$A\left(\frac{2}{5}\right) = 2 \quad \to \quad A = 5$$

As a result, the function that satisfies all the conditions is

$$f(x) = 5\frac{(x-1)(x+1)}{(x-4)(x+1)}.$$

Evaluate the function at $x = 0$.

$$f(0) = 5\frac{(-1)(1)}{(-4)(1)} = 5\left(\frac{1}{4}\right) = \frac{5}{4}$$

Evaluate the limit of the function as $x \to \infty$.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} 5\frac{(x-1)(x+1)}{(x-4)(x+1)} \\
&= 5 \lim_{x\to\infty} \frac{x^2-1}{x^2-3x-4} \\
&= 5 \lim_{x\to\infty} \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1-\frac{3}{x}-\frac{4}{x^2}\right)} \\
&= 5 \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1-\frac{3}{x}-\frac{4}{x^2}} \\
&= 5\left(\frac{1-0}{1-0-0}\right) \\
&= 5(1) \\
&= 5
\end{aligned}$$